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STATIONARY DIFFRACTION OF WAVES BY A CIRCULAR APERTURE IN AN ELASTIC HALF-PLANE *

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A solution of the problem of the stationary diffraction of waves by a circular aperture in an elastic half-plane is suggested. It is obtained by using the methods of separation of variables, re-expansion of cylindrical functions in plane waves and multiple reflections. A solution for a different type of incident wave is constructed.

A solution to the similar problem of antiplane deformation was obtained using the method of reflected sources /1/. When there is plane deformation the problem of steady-state oscillations in /2/ is reduced to a system of Fredholm integral equations of the second kind.

1. Formulation of the problem. Consider the elastic (λ, μ, ρ) isotropic semi-space $x < h$, $h > 0$ with a circular cylindrical cavity of radius R, whose OZ axis is parallel to the boundary of the half-space (Fig.1). Suppose a stationary load acts on the boundary of the cavity

$$\sigma_{rr} = p_r(\theta) e^{-i\omega t}, \tag{1.1}$$
$$\sigma_{r\theta} = p_\theta(\theta) e^{-i\omega t}, \quad \sigma_{rz} = 0$$

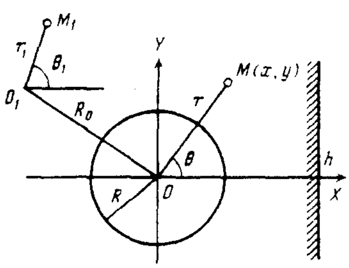

Fig.1

where p_j $(j = r, \theta)$ are the known functions which can be expanded in a complex Fourier series

$$p_j = \sum_n p_{jn} e^{in\theta}, \quad j = r, \theta .$$

Here r, θ, z is a cylindrical set of coordinates with the polar axis OX, and σ_{ij} are the components of the stress tensor.

The boundary of the half-space is not affected by the loads

$$\sigma_{xx} = \sigma_{xy} = \sigma_{xz} = 0, \quad x = h. \tag{1.2}$$

Under conditions (1.1) and (1.2) a state of plane deformation is realized which can be described by the Lamé potentials φ, ψ, satisfying the Helmholtz equations /3, 4/

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \alpha^2\varphi = 0, \quad \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \beta^2\psi = 0 . \tag{1.3}$$

The displacement components u_x, u_y are defined in terms of φ, ψ by the relations

$$u_x = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi}{\partial y}, \quad u_y = \frac{\partial\varphi}{\partial y} - \frac{\partial\psi}{\partial x} . \tag{1.4}$$

Here and everywhere below we omit the time multiplier $e^{-i\omega t}$, and determine the complex amplitudes $\alpha = \omega\sqrt{\rho/(\lambda + 2\mu)}$, $\beta = \sqrt{\rho/\mu}\cdot\omega$.

Using Hooke's law for an elastic medium /4/, we write the boundary conditions for the potentials

$$\mathbf{L}\varphi - \mathbf{M}\psi = -\frac{p_r(\theta)}{2\mu}, \quad \mathbf{M}\varphi - \mathbf{L}\psi = -\frac{p_\theta(\theta)}{2\mu} \tag{1.5}$$
$$\mathbf{L} = \frac{1}{2}\beta^2 + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2},$$
$$\mathbf{M} = \frac{1}{r^2}\frac{\partial}{\partial\theta} - \frac{1}{r}\frac{\partial^2}{\partial r\partial\theta}, \quad r = R$$

$$\left(\alpha^2 - 0.5\beta^2 + \frac{\partial^2}{\partial x^2}\right)\varphi + \frac{\partial^2\psi}{\partial x\,\partial y} = 0 \tag{1.6}$$
$$\frac{\partial^2\varphi}{\partial x\,\partial y} - \left(\beta^2 + \frac{\partial^2}{\partial x^2}\right)\psi = 0, \quad x = h .$$

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We separate the potentials of the unknown field into components, each of which appropriately satisfies from Eqs. (1.3)

$$\varphi=\sum_{j=0}^{\infty}\varphi^j,\quad \psi=i\sum_{j=0}^{\infty}\psi^j. \tag{1.7}$$

We conditionally call the potentials with even indices waves which travel from the cavity, and we call those with odd indices reflections from the plane boundary of the half-space. The displacement and stress components u_k^j, σ_{kl}^j which correspond to them satisfy the following conditions on the boundary of the domain:

$$\sigma_{xj}^{2k+1}=-\sigma_{xj}^{2k},\ j=x,\ y,\ x=h \tag{1.8}$$

$$\sigma_{rj}^{2k+2}=-\sigma_{rj}^{2k+1},\ j=r,\ \theta,\ r=R;\ k=0,1,2,\ldots. \tag{1.9}$$

Relations (1.1) hold for σ_{rr}^0, $\sigma_{r\theta}^0$. It follows from (1.8) and (1.9) that the total potentials (1.7) satisfy the boundary conditions (1.5) and (1.6).

We shall seek the even potentials in the form of the Fourier-Bessel series

$$\varphi^j=\sum_{n=-\infty}^{\infty}a_n^jH_n^{(1)}(\alpha r)e^{in\theta},\quad \psi^j=\sum_{n=-\infty}^{\infty}b_n^jH_n(\beta r)e^{in\theta} \tag{1.10}$$

where $H_n^{(1)}(\cdot)$ is a Hankel function which satisfies Sommerfeld's radiation conditions /1, 4/. The components in the sum (1.7) are partial solutions of Helmholtz's Eqs. (1.3). We shall represent the potentials of the waves reflected from the boundary of the half-plane in the form

$$\varphi^j=\frac{1}{i\pi}\int_{L_\varepsilon}\frac{a^j(\xi)}{\Delta(\xi)}\exp\left(iy\xi-\sqrt{\xi^2-\alpha^2}(x-h)\right)d\xi \tag{1.11}$$

$$\psi^j=\frac{1}{i\pi}\int_{L_\varepsilon}\frac{b^j(\xi)}{\Delta(\xi)}\exp\left(iy\xi-\sqrt{\xi^2-\beta^2}(x-h)\right)d\xi$$

$$\operatorname{Re}\sqrt{\xi^2-k^2}\geqslant 0,\ \operatorname{Im}\sqrt{\xi^2-k^2}\leqslant 0,\ k=\alpha,\beta$$

$$\Delta(\xi)=4\xi^2\sqrt{\xi^2-\alpha^2}\sqrt{\xi^2-\beta^2}-(2\xi^2-\beta^2)^2=0. \tag{1.12}$$

Here L_ε is a contour in the plane of the complex variable $\xi=\xi_1+i\xi_2$, and coincides with the ξ_1 axis everywhere except in the ε-vicinity of the points $\pm\alpha,\pm\beta,\pm\gamma$, where $\pm\gamma$ are two real roots of Rayleigh's equation, $\alpha<\beta<\gamma$ [4, 5], which must be circumvented with respect to the ε-semicircles, as shown in Fig. 2. The conditions on the radical signs in (1.11) correspond to the waves which leave the boundary and decay as $x\to\infty$.

The function $\sqrt{\xi^2-k^2}$ $(k=\alpha,\beta)$ has branching points $\pm k$. The sections of the ξ plane along the ξ_2 axis and the segment $|\xi_1|<k$, on which $\operatorname{Re}\sqrt{\xi^2-k^2}=0$, separate the two sheets of the Riemann surface of the function $\sqrt{\xi^2-k^2}$, in each of which the quantity $\operatorname{Re}\sqrt{\xi^2-k^2}$ is either positive or negative. Fig. 2 shows the domain of $\operatorname{Im}\sqrt{\xi^2-k^2}$, which is of fixed sign, on the sheet $\operatorname{Re}\sqrt{\xi^2-k^2}\geqslant 0$. The contour L_ε must pass into the domain $\operatorname{Im}\sqrt{\xi^2-k^2}\leqslant 0$, which also dictates the above-mentioned direction of the circuits of the points $\pm\alpha$, $\pm\beta$, $\pm\gamma$.

The boundary conditions (1.1), (1.5) and (1.6) enable us successively to determine φ^j,ψ^j, if φ^{j-1},ψ^{j-1} are known.

2. A lumped source in the half-plane.

Suppose φ^{j-1},ψ^{j-1} are known and are defined by the relations (1.10). To find φ^j,ψ^j we change the potentials φ^{j-1},ψ^{j-1} represented by cylindrical functions - expanding them into plane waves. For this we use the representation /6/

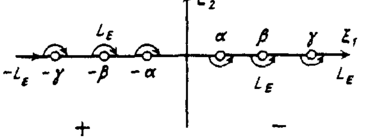

Fig.2

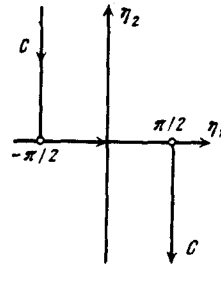

Fig.3

$$H_n^{(1)}(kr)e^{in\theta} = \frac{(-i)^n}{\pi}\int_C \exp(ik(x\cos\eta + y\sin\eta) + in\eta)\,d\eta \qquad (2.1)$$

$$x > 0$$

(the countour C is shown in Fig.3). We introduce the change of variables $\xi = \xi_1 + i\xi_2 = k\sin\eta$, converting the integration contour C into the real axis $L: \xi_2 = 0$. Along C

$$\cos\eta = i\sqrt{\xi^2 - k^2}/k, \quad e^{in\eta} = i^n(\xi + \sqrt{\xi^2 - k^2})^n/k^n$$

$$\operatorname{Re}\sqrt{\xi^2 - k^2} \geqslant 0, \ \operatorname{Im}\sqrt{\xi^2 - k^2} \leqslant 0.$$

Hence it follows that

$$H_n^{(1)}(kr)e^{in\theta} = \frac{1}{i\pi}\int_L \left(\frac{\xi + \sqrt{\xi^2 - k^2}}{k}\right)^n \frac{\exp(iy\xi - x\sqrt{\xi^2 - k^2})}{\sqrt{\xi^2 - k^2}}\,d\xi \qquad (2.2)$$

$$x > 0$$

Consider the contour $C_\varepsilon^k: \eta = \operatorname{Arcsin}(\xi/k)$ $(k = \alpha, \beta)$, $|\eta_1| < \pi/2$, where $\xi \in L_\varepsilon$. By virtue of the analyticity of the integrand in (2.1) the contour C_ε^k is equivalent to C; therefore in the representation (2.2) we can change the integration contour L to L_ε.

Using (2.2) for $\varphi^{j-1}, \psi^{j-1}$ $(j = 2m + 1)$, determined by the series (1.10) and (1.11) for the potentials φ^j, ψ^j, we shall write the boundary conditions (1.6), which when $x = h$ have the form of Fourier integrals with correction on the contour L_ε. Equating coefficients of $e^{iy\xi}$, we obtain a linear set of equations for determining the functions $a^j(\xi), b^j(\xi)$. Solving it, we obtain

$$a^j(\xi) = g_1(\xi, \alpha)\Sigma_a - g_2(\xi)\Sigma_b, \ b^j(\xi) = g_2(\xi)\Sigma_a + g_1(\xi)\Sigma_b \qquad (2.3)$$

$$\Sigma_a = \sum_{n=-\infty}^{\infty} a_n^j f_n(\xi, \alpha), \quad \Sigma_b = \sum_{n=-\infty}^{\infty} b_n^j f_n(\xi, \beta)$$

$$g_1(\xi, k) = \frac{4\xi^2\sqrt{\xi^2 - \alpha^2}\sqrt{\xi^2 - \beta^2} + (2\xi^2 - \beta^2)^2}{\sqrt{\xi^2 - k^2}}, \quad g_2(\xi) = 4\xi(2\xi^2 - \beta^2)$$

$$f_n(\xi, k) = \left(\frac{\xi + \sqrt{\xi^2 - k^2}}{k}\right)^n \exp(-h\sqrt{\xi^2 - k^2}).$$

3. Diffraction at an aperture.

To determine the functions $\varphi^{j-1}, \psi^{j-1}$ $(j = 2k + 1)$, we will change to a polar system of coordinates in expressions (1.11). Bearing in mind the expansion /3/

$$e^{ikr\cos\eta} = \sum_{n=-\infty}^{\infty} i^n J_n(kr)e^{in\eta} \qquad (3.1)$$

which is analytically continuable into the plane of complex η, we can expand the potentials (1.11) using the cylindrical functions

$$\varphi^j = \frac{1}{i\pi}\sum_n J_n(\alpha r)e^{in\theta}\int_{L_\varepsilon}\frac{a^j(\xi)}{\Delta(\xi)} f_n(\xi, \alpha)E_\alpha(\xi)\,d\xi = \sum_{n=-\infty}^{\infty} a_n^j J_n(\alpha r)e^{in\theta} \qquad (3.2)$$

$$\psi^j = \frac{1}{i\pi}\sum_n J_n(\beta r)e^{in\theta}\int_{L_\varepsilon}\frac{b^j(\xi)}{\Delta(\xi)} f_n(\xi, \beta)E_\beta(\xi)\,d\xi = \sum_{n=-\infty}^{\infty} b_n^j J_n(\beta r)e^{in\theta}$$

$$E_k(\xi) = \exp(-h\sqrt{\xi^2 - k^2}).$$

From the above and from (2.3) it follows that

$$i\pi a_n^j = \sum_{m=-\infty}^{\infty} a_m^{j-1}\int_{L_\varepsilon}\frac{g_1(\xi, \alpha)}{\Delta(\xi)} f_{n+m}(\xi, \alpha)E_\alpha(\xi)\,d\xi - \sum_{m=-\infty}^{\infty} b_m^{j-1}\int_{L_\varepsilon}\frac{g_2(\xi)}{\Delta(\xi)} f_m(\xi, \beta)f_n(\xi, \alpha)E_\alpha(\xi)\,d\xi \qquad (3.3)$$

$$i\pi b_n^j = \sum_{m=-\infty}^{\infty} b_m^{j-1}\int_{L_\varepsilon}\frac{g_1(\xi, \beta)}{\Delta(\xi)} f_{n+m}(\xi, \beta)E_\beta(\xi)\,d\xi +$$

$$\sum_{m=-\infty}^{\infty} a_m^{j-1} \int\limits_{L_\varepsilon} \frac{g_2(\xi)}{\Delta(\xi)} f_m(\xi,\alpha) f_n(\xi,\beta) E_\beta(\xi)\, d\xi .$$

The determination of the functions $\varphi^{j+1}, \psi^{j+1}$ reduces to the solution of the problem of the stationary diffraction of waves (3.2) by a circular aperture in an infinite plane, whose solution is well known /1/. In the notation used here

$$\begin{aligned}
&a_n^{j+1} = (a_n{}^j\Delta_{n1}(\alpha R, \beta R) + b_n{}^j\Delta_{n2}(\beta R))/\Delta_n \\
&b_n^{j+1} = (a_n{}^j\Delta_{n2}(\alpha R) + b_n{}^j\Delta_{n1}(\beta R, \beta R))/\Delta_n \\
&\Delta_n = \Phi_{1n}(\alpha R, H_n^{(1)})\Phi_{1n}(\beta R, H_n^{(1)}) - \Phi_{2n}(\alpha R, H_n^{(1)})\Phi_{2n}(\beta R, H_n^{(1)}) \\
&\Delta_{n1}(z_1, z_2) = \Phi_{2n}(z_1, J_n)\Phi_{2n}(z_2, H_n^{(1)}) - \Phi_{1n}(z_1, J_n)\Phi_{1n}(z_2, H_n^{(1)}) \\
&\Delta_{n2}(z) = \Phi_{1n}(z, J_n)\Phi_{2n}(z, H_n^{(1)}) - \Phi_{2n}(z, J_n)\Phi_{1n}(z, H_n^{(1)}) \\
&\Phi_{1n}(z, Z_n) = (2n^2 - \beta^2R^2)Z_n(z) - 2zZ_n'(z), \\
&\Phi_{2n}(z, Z_n) = 2n(Z_n(z) - zZ_n'(z)) .
\end{aligned} \tag{3.4}$$

Here Z_n is the identifier of the cylindrical function $H_n^{(1)}$ or J_n. When $j = 0$

$$\begin{aligned}
&a_n{}^0 = (p_{n\theta}\Phi_{2n}(\beta R, H_n^{(1)}) - p_{nr}\Phi_{1n}(\beta R, H_n^{(1)}))/\Delta_n \\
&b_n{}^0 = (p_{nr}\Phi_{2n}(\alpha R, H_n^{(1)}) - p_{n\theta}\Phi_{1n}(\alpha R, H_n^{(1)}))/\Delta_n .
\end{aligned} \tag{3.5}$$

The process of solving the problem thus consists of the following. Using formulas (3.5) we find $a_n{}^0, b_n{}^0$, i.e. the potential of the first wave emitted by the cavity. Then, using (2.3), (3.2) are the potentials of the waves reflected from the boundary of the half-plane. Further, using formulas (3.4), we determine $a_n{}^2, b_n{}^2$ etc.

We can represent the solution by cylindrical functions, using for φ^j, ψ^j either relations (1.10) and (3.2) or the Fourier-type contour integrals (1.11) and (2.2) when $0 \leqslant x \leqslant h$. A suitable representation is chosen, dependent on the boundary in whose vicinity the characteristics of the stress-deformed state u_j, σ_{ij} are chosen. In view of their awkwardness, the expressions for the latter are not presented here.

In relations (1.11), (3.2) and (3.3) we can pass to the limit as $\varepsilon \to 0$, as in /7/. The integrals obtained should then be understood in the sense of the principal value in the vicinity of the points $\pm\gamma$, around $\pm\alpha, \pm\beta$, and they exist as non-eigenvalues. In addition, terms appearing outside the integral will occur which, apart from a multiplier, are equal to the difference of the residues of the integrands at the points $\pm\gamma$, which describe the effect of Rayleigh surface wave.

We can propose other methods of calculating the integrals along L_ε, based on the transformation of the integration contour using Jordan's lemma, modifying it for a domain with branching points; we shall not dwell on them here.

4. Investigation of the convergence of the series and the existence of the integrals. Using the asymptotic form of Hankel's functions when $|n| \to \infty$ and fixed argument /8/, from relations (3.5) we obtain

$$\begin{aligned}
&|a_n{}^0| \sim \frac{C\pi}{2}\frac{|p_{nr} + p_{n\theta}|}{(|n|-1)!}\left(\frac{\alpha R}{2}\right)^{|n|} \\
&|b_n{}^0| \sim \frac{C\pi}{2}\frac{|p_{nr} + p_{n\theta}|}{(|n|-1)!}\left(\frac{\beta R}{2}\right)^{|n|}, \quad C = \frac{R}{\mu\beta} .
\end{aligned}$$

Hence for fairly large $|n|$ it follows that

$$\begin{aligned}
&|a_n{}^0H_n^{(1)}(\alpha R)| < C\,|p_{nr} + p_{n\theta}|\left(\frac{R}{r}\right)^{|n|} < C\,|p_{nr} + p_{n\theta}| \\
&|b_n{}^0H_n^{(1)}(\beta R)| < C\,|p_{nr} + p_{n\theta}|
\end{aligned}$$

i.e. series (1.10) converge absolutely and uniformly with respect to r, θ and are continuous in the domain $r \geqslant R$ of the function. If $|p_{nj}| = o(|n|^{-4})$ is required when $|n| \to \infty$ $(j = r, \theta)$, we can obtain similar estimates for the differentiated series, which validates the convergence and operation of term-by-term differentiation of series (1.10). For $j = 2k$, $k = 1, 2, \ldots$ the convergence of the Fourier-Bessel series is similarly proved on the basis of the existence of continuous and differentiable potentials $\varphi^{j-1}, \psi^{j-1}$.

Note that, by virtue of the choice of the integration contour L_ε, the integrands in expressions (1.11) are infinitely differentiable with respect to x and y, and approach zero as $|\xi| \to \infty$. Since

$$|f_n(\xi, k)| < (2|\xi|/k)^{|n|} \text{ when } |\xi| > k, \quad |f_n(\xi, k)| = 1 \text{ when } |\xi| \leqslant k$$

for fairly large N we have

$$\Big|\sum_{|n|>N} a_n^0 f_n(\xi, \alpha)\Big| < C \sum_{|n|>N} |p_{nr} + p_{n\theta}|\, |R\xi|^{|n|}/(|n|-1)! \leqslant C \sum_{|n|>N} |R\xi|^{|n|}/|n|! \leqslant 2Ce^{R|\xi|} .$$

Since for fairly large $|\xi|$

$$\Big|\sum_{|n|>N} a_n^0 f_n(\xi, \alpha) \exp((x-h)\sqrt{\xi^2-\alpha^2}) \leqslant 2C\exp((x-2h+R)|\xi|) \leqslant$$
$$2C\exp((R-h)|\xi|)$$
$$\Big|\sum_{|n|<N} a_n^0 f_n(\xi, \alpha) \exp((x-h)\sqrt{\xi^2-\alpha^2})\Big| \leqslant$$
$$2\sum_{|n|<N} |a_n^0|(2|\xi|/\alpha)^{|n|} e^{-h|\xi|}$$

and $R < h$, $h > 0$, the integrals (1.11) exist and are continuous functions with respect to x and y in the domain $x < h$.

Similar estimates can be obtained for the integrals (1.11), formally differentiated with respect to x and y, whose uniform convergence follows from the existence of the integrand majorant which can be integrated in $(-\infty, \infty)$

$$|\xi|^m \exp((R-h)|\xi|), \quad |\xi|^m \exp(-h|\xi|)$$

which also validates the differentiation operation under the sign of the integral.

Consider the problem of the convergence of the method of successive reflections, i.e. the series (1.7). Note that the solution of the problem can be reduced to a solution of the infinite set of linear equations with a determinant of the normal type. The procedure described above in an implementation of the method of successive approximations. For this the unknown potentials should be represented in the form $\varphi = \varphi_0 + \varphi_1$, $\psi = \psi_0 + \psi_1$ preserving the previous form (1.10) for φ_0, ψ_0 and (1.11) for φ_1, ψ_1. a_n^1, b_n^1 are determined in terms of a_n^0, b_n^0 from the relations (3.3). From the conditions on the contour of the cavity (1.5) equating the coefficients for $e^{in\theta}$ - we obtain an infinite set of linear equations for determining the coefficients a_n^0, b_n^0.

By virtue of the peculiarities of the behaviour of Hankel functions as $n \to \infty$, the system obtained is not suitable for analysis. If, instead of the coefficients a_n^0, b_n^0 we introduce

$$c_n^0 = a_n^0 \frac{(|n|+1)^3 |2n|!!}{(\alpha R)^{|n|}}, \quad d_n^0 = b_n^0 \frac{(|n|+1)^3\, |2n|!!}{(\beta R)^{|n|}}$$

and write the set with the corresponding change a_n^0, b_n^0, we shall arrive at a system with a determinant of the normal type. The sum of the moduli of the coefficients of the matrix of the system is limited by the double series

$$C \sum_{n=0}^{\infty} (1-n)^2 \left(\frac{R}{2h-R}\right)^n - \frac{(n-1)(\beta R)^{2n}}{(2n)!!} \sum_{p=0}^{\infty} \frac{(p-1)!\,(p-1)^4}{(p+n)!} \left(\frac{R}{2h}\right)^p$$

whose convergence follows from the condition $R < h$. The free terms approach zero as $|n| \to \infty$ no slower than $(|p_{nr} + p_{n\theta}|)n^4 \to 0$. Thus the conditions of the existence and uniqueness of the bounded solution (see /9/) - which can be obtained using the reduction method or the method of successive approximations, as described above - hold.

It follows from the latter inequality that the deeper the aperture, the more rapid the convergence and the lower the frequency of the acting load. From the physical point of view, the method of successive reflections is preferable since often, if great calculational accuracy is not required, we can limit ourselves to one or two reflections to obtain data which are authentic from an engineering point of view.

5. Problems of stationary diffraction.

When solving problems of stationary diffraction the field of the incident wave is usually considered as given. Any solutions of Eqs. (1.3) are taken as the field. Here by the incident wave - whose potentials we denote by Φ_+, Ψ_+ - we shall understand the different solutions for a continuous elastic half-space with a free boundary (1.6). The solution of the diffraction problem will be sought in the form

$$\varphi = \Phi_+ + \Phi_-, \quad \psi = \Psi_+ + \Psi_-$$

where Φ_-, Ψ_- are potentials of the waves reflected from the free boundary. It is clear that we can reduce the problem of determining Φ_-, Ψ_- to that considered above.

In seismology, when analyzing the influence of remote seismic waves on buildings, we usually consider the plane longitudinal and transverse waves and Rayleigh surface waves. For close earth tremors or oscillations of artificial origin, suchas industrial explosions, it is convenient to model the field of the incident wave as the field of a lumped source (the centre of pressure, concentrated momenta, forces, dipoles, etc.). We shall assume that in the vicinity of the cavity these fields can be expanded as follows:

$$\Phi_+ = \sum_{n=-\infty}^{\infty} a_1^n J_n(\alpha r) e^{in\theta}, \quad \Psi_+ = i \sum_n b_n^1 J_n(\beta r) e^{in\theta} . \tag{5.1}$$

We shall present some of them.

Plane longitudinal harmonic waves

$$\Phi_+ = \exp(i\alpha\rho\cos(\eta-\beta)) + \frac{d_1(\xi)}{\Delta(\xi)} \exp(-i\alpha\rho\cos(\eta+\delta))$$
$$\Psi_+ = \frac{d_2(\xi,\beta)}{\Delta(\xi)} \exp(-i\beta\rho\cos(\eta+\delta^*))$$
$$i^n a_n^1 = (-1)^n(-i(\alpha h\cos\delta + n\delta) + \frac{d_1(\xi)}{\Delta(\xi)} \exp(i(\alpha h\cos\delta + n\delta))$$
$$i^n b_n^1 = (-1)^n \frac{d_2(\xi,\alpha)}{\Delta(\xi)} \exp(i(\beta h\cos\delta^* + n\delta^*)) .$$

Here δ is the angle between the wave vector and OX axis.

$$\xi = \alpha\sin\delta, \quad \delta^* = \arcsin(\xi/\beta), \quad \rho\cos\eta = x - h, \quad \rho\sin\eta = y .$$

Plane transverse waves

$$\Phi_+ = \frac{d_2(\xi,\beta)}{\Delta(\xi)} \exp(-i\alpha\rho\cos(\eta+\delta^*))$$
$$\Psi_+ = \exp(i\beta\rho\cos(\eta-\delta)) + \frac{d_1(\xi)}{\Delta(\xi)} \exp(-i\beta\rho\cos(\eta+\delta))$$
$$i^n a_n^1 = \frac{d_2(\xi,\beta)}{\Delta(\xi)} \exp(i(\alpha h\cos\delta^* + n\delta^*))$$
$$i^n b_n^1 = (-1)^n \exp(-i(\beta h\cos\delta + n\delta)) +$$
$$\frac{d_1(\xi)}{\Delta(\xi)} \exp(i(\beta h\cos\delta + n\delta))$$
$$\xi = \beta\sin\delta, \quad \delta^* = \mathrm{Arc}\sin(\xi/\alpha) .$$

Rayleigh waves

$$\Phi_+ = \exp(i\gamma y + (x-h)\sqrt{\gamma^2-\alpha^2}),$$
$$\Psi_+ = \frac{2i\gamma\sqrt{\gamma^2-\alpha^2}}{2\gamma^2-\beta^2} \exp(i\gamma y + \sqrt{\gamma^2-\beta^2}(x-h))$$
$$a_n^1 = f_n(\gamma,\alpha)\exp(-h\sqrt{\gamma^2-\alpha^2}),$$
$$b_n^1 = \frac{2\gamma\sqrt{\gamma^2-\alpha^2}}{2\gamma^2-\beta^2} f_n(\gamma,\beta)\exp(-h\sqrt{\gamma^2-\beta^2})$$
$$(d_i(\xi) = g_i(\xi,k)\sqrt{\xi^2-k^2}, \; i = 1, 2) .$$

Waves from a lumped source. We can represent the field of the source, taking account of its reflection from the plane boundary, in the form

$$\Phi_+ = \sum_n (A_n^0 H_n^{(1)}(\alpha r_1) + A_n^1 J_n(\alpha r_1)) \exp(in\theta_1)$$
$$\Psi_+ = \sum_n (B_n^0 H_n^{(1)}(\beta r) + B_n^1 J_n(\beta r_1)) \exp(in\theta_1) .$$

Here r_1, θ_1 (Fig.1) are polar coordinates connected to the source. The coefficients A_n^1, B_n^1 are calculated using formulas (3.3), with a_m^{j-1}, b_m^{j-1} replaced by A_m^0, B_m^0 and a_n^j, b_n^j by A_n^1, B_n^1.

Using the summation theorem for Bessel functions /1/, we can change to the form (5.1). As a result we obtain

$$a_n^1 = \sum_{p=-\infty}^{\infty} (A_{n+p}^1 J_p(\alpha R_0) + A_{n+p}^0 H_p^{(1)}(\alpha R_0)) \exp(ip\theta_0)$$
$$b_n^1 = \sum_{p=-\infty}^{\infty} (B_{n+p}^1 J_p(\beta R_0) + B_{n+p}^0 H_p^{(1)}(\beta R_0)) \exp(ip\theta_0) .$$

(R_0, θ_0) are the polar coordinates of the point O in the system (O_1, r_1, θ_1).

The field of the reflected waves has the form

$$\Phi_- = \sum_{j=2}^{\infty} \varphi_j, \quad \Psi_- = \sum_{j=2}^{\infty} \psi_j .$$

Since the coefficients a_n^1, b_n^1 are known, the component potentials are determined successively, as described above.

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REFRACTION OF PLANE-POLARIZED WAVES AT THE BOUNDARY OF AN ELASTIC AND ELASTOPLASTIC HALF-SPACE*

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Selfsimilar solutions of dynamic equations for antiplane deformation in an ideal elasto plastic medium are considered. A solution is constructed of the problem of the refraction of plane-polarized plane waves of an arbitrary profile which penetrate from the elastic to the elastoplastic half-space.

Selfsimilar solutions were investigated earlier /1-4/ when the rates of displacements and stresses depend only on the ratio of the coordinates. The selfsimilar problem of the refraction of a plane elastic wave into an elastoplastic half-space with boundary conditions like those of Coulomb's law of dry friction, and conditions guaranteeing full contact at the boundary of separation, were analysed in /5, 6/.

1. Consider the dynamic problem of the theory of complex displacement in an ideal elastoplastic medium. In a rectangular Cartesian system of coordinates x_i the vector of the rate of displacement w is directed along x_3 axis and depends only on x_1, x_2 and the time t.

All the components of stress vanish, apart from $\tau_1 = \sigma_{13}(x_1, x_2, t)$, $\tau_2 = \sigma_{23}(x_1, x_2, t)$. The equations of motion in this case have the form

$$\frac{\partial \tau_1}{\partial x_1} + \frac{\partial \tau_2}{\partial x_2} - \rho \frac{\partial w}{\partial t} = 0 . \tag{1.1}$$

The full deformations are the sum of the elastic and the plastic part, and the elastic deformations are connected with the stresses by Hooke's law

$$\gamma_1 = \gamma_1^e + \gamma_1^p, \quad \gamma_2 = \gamma_2^e + \gamma_2^p; \quad \tau_1 = 2\mu\gamma_1^e, \quad \tau_2 = 2\mu\gamma_2^e . \tag{1.2}$$

In the plastic domain, the stresses satisfy the condition of plasticity, and the rates of the plastic deformations are determined from the associated flow rule

$$\tau_1^2 + \tau_2^2 = k^2; \ \gamma_1^{\cdot p} = \psi\tau_1, \ \gamma_2^{\cdot p} = \psi\tau_2 . \tag{1.3}$$

The total rates of deformation are expressed in terms of the displacements by

$$\gamma_1^{\cdot} = \frac{1}{2}\frac{\partial w}{\partial x_1}, \quad \gamma_2^{\cdot} = \frac{1}{2}\frac{\partial w}{\partial x_2} . \tag{1.4}$$

Differentiating relations (1.2) with respect to time and eliminating the values of the

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